Chapter 11

Rolling, Torque, and Angular Momentum

Hyperphysics: Circular Motion and Rotation

11.2 Rolling as Translational and Rotation Combined

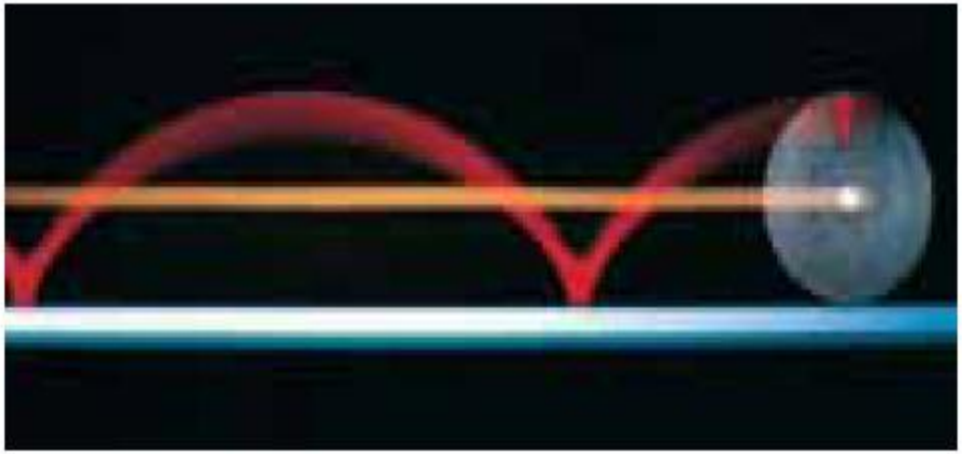


Fig. 11-2 A time-exposure photograph of a rolling disk. Small lights have been attached to the disk, one at its center and one at its edge. The latter traces out a curve called a *cycloid*. *(Richard Megna/Fundamental Photographs)*

- Although the center of the object moves in a straight line parallel to the surface, a point on the rim certainly does not.

- This motion can be studied by treating it as a combination of translation of the center of mass and rotation of the rest of the object around that center.

11.2 Rolling

$$s = \theta R, \quad \longrightarrow \quad v_{com} = \omega R \qquad \text{(smooth rolling motion).}$$

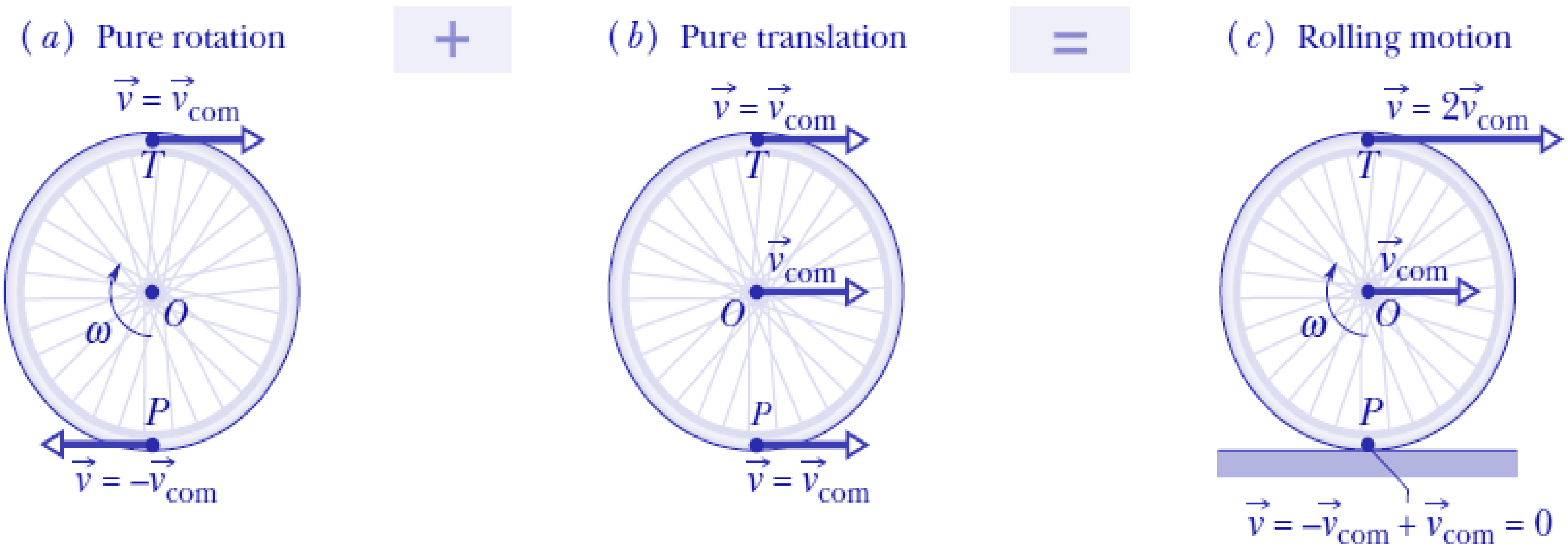


Fig. 11-4 Rolling motion of a wheel as a combination of purely rotational motion and purely translational motion. (*a*) The purely rotational motion: All points on the wheel move with the same angular speed ω. Points on the outside edge of the wheel all move with the same linear speed $v = v_{com}$. The linear velocities $\vec{v}$ of two such points, at top (T) and bottom (P) of the wheel, are shown. (*b*) The purely translational motion: All points on the wheel move to the right with the same linear velocity $\vec{v}_{com}$. (*c*) The rolling motion of the wheel is the combination of (*a*) and (*b*).

11.3 The Kinetic Energy of Rolling

- If we view the rolling as pure rotation about an axis through P, then:

$$K = \tfrac{1}{2} I_P \omega^2,$$

- *ω is the angular speed of the wheel; I_P is the rotational inertia of the wheel about the axis through P.*

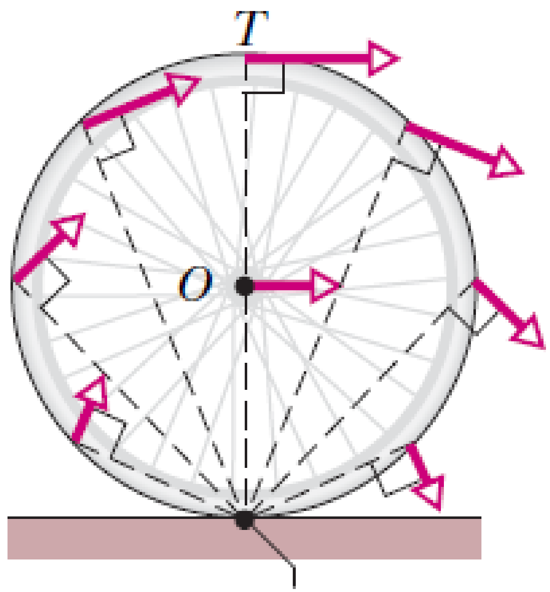
- Using the parallel-axis theorem *($I_P = I_{com} + Mh^2$):*

$$I_P = I_{\mathrm{com}} + MR^2$$

- *M is the mass of the wheel, I_{com} is its rotational inertia about an axis through its center of mass, and R is the wheel's radius, at a perpendicular distance h.*

- Using the relation $v_{com} = \omega R$, we get:

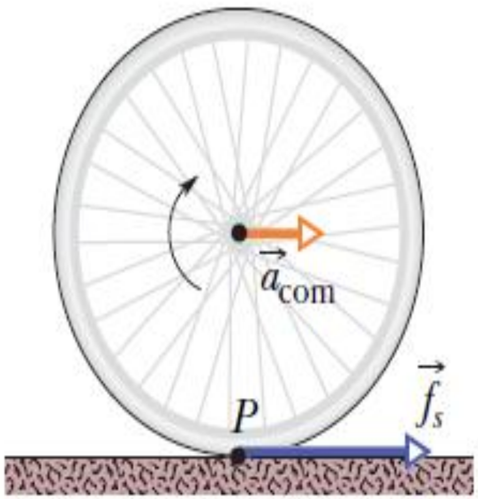
$$K = \tfrac{1}{2} I_{\mathrm{com}} \omega^2 + \tfrac{1}{2} M v_{\mathrm{com}}^2.$$



A rolling object, therefore has two types of kinetic energy:

1. **A rotational kinetic energy due to rotation about the center of mass (= ½ $I_{com}\omega^2$);**
2. **A translational kinetic energy due to translation of the center of mass (= ½ Mv^2_{com})**

11.4: The Forces of Rolling: Friction and Rolling



A wheel rolls horizontally without sliding while accelerating with linear acceleration a_{com}. A static frictional force f_s acts on the wheel at P, opposing its tendency to slide.

The magnitudes of the linear acceleration a_{com}, and the angular acceleration a can be related by:

$$a_{com} = \alpha R \qquad \text{(smooth rolling motion)}$$

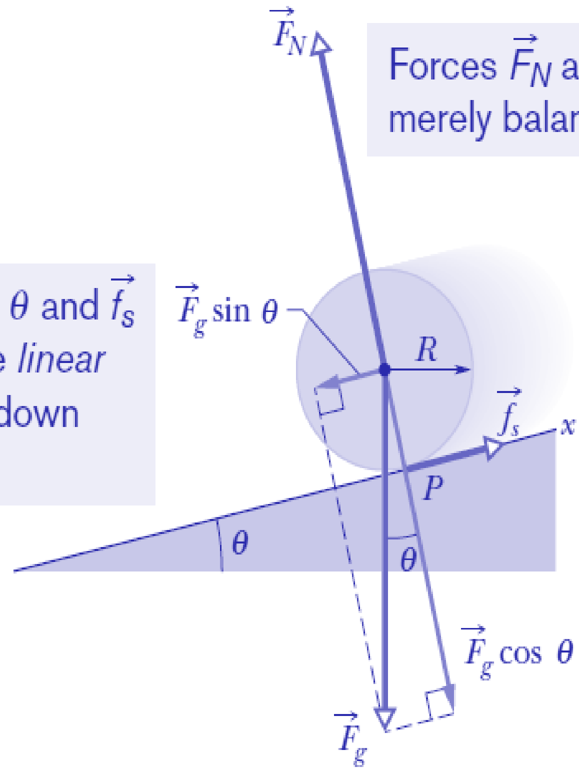
where R is the radius of the wheel.

If the wheel does slide when the net force acts on it, the frictional force that acts at P in *Fig. 11-3* is a kinetic frictional force, f_k .The motion then is not smooth rolling, and the above relation does not apply to the motion.

11.4: The Forces of Rolling: Rolling Down a Ramp

Forces $\vec{F}_N$ and $\vec{F}_g \cos\theta$ merely balance.

Forces $\vec{F}_g \sin\theta$ and $\vec{f}_s$ determine the *linear* acceleration down the ramp.



The torque due to $\vec{f}_s$ determines the *angular* acceleration around the com.

A round uniform body of radius R rolls down a ramp. The forces that act on it are the gravitational force F_g, a normal force F_N, and a frictional force f_s pointing up the ramp.

$$\tau_{net} = I\alpha \longrightarrow Rf_s = I_{com}\alpha$$

$$a_{com} = \alpha R \longrightarrow f_s = -I_{com}\frac{a_{com,x}}{R^2}$$

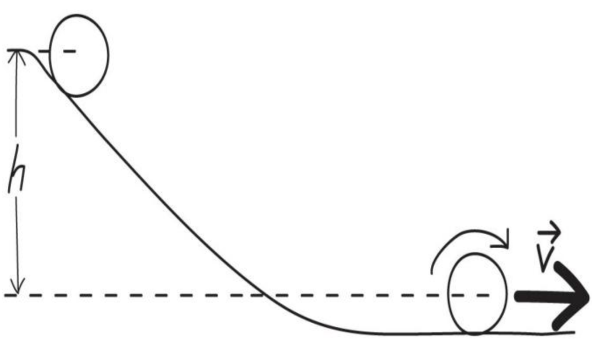
$$a_{com,x} = -\frac{g\sin\theta}{1 + I_{com}/MR^2}$$

11.4: The Forces of Rolling: Rolling Down a Ramp

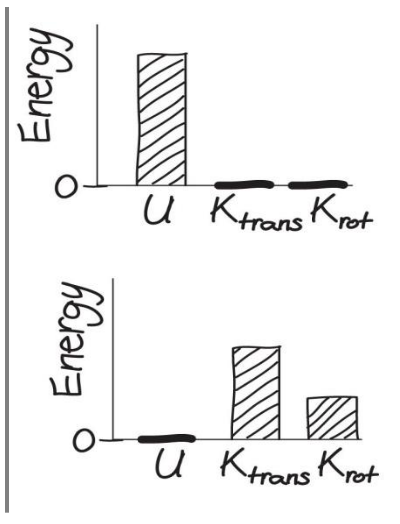
- A rotating object has kinetic energy $K_{\text{rot}} = \frac{1}{2} I\omega^2$ associated with its rotational motion alone.
  - It may also have translational kinetic energy: $K_{\text{trans}} = \frac{1}{2} Mv^2$.

- In problems involving energy conservation with rotating objects, both forms of kinetic energy must be considered.
  - For rolling objects, the two are related:
  - The relation depends on the rotational inertia.

A solid ball rolls down a hill. How fast is it moving at the bottom?



Energy bar graphs



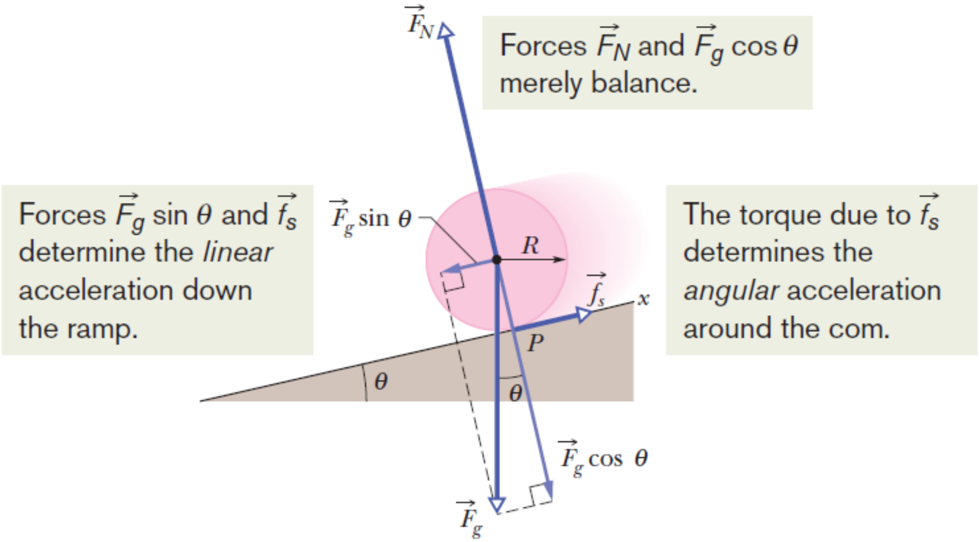
Equation for energy conservation:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{7}{10}Mv^2$$

Solution:

$$v = \sqrt{\frac{10}{7}gh}$$

Example: Rolling Down a Ramp



A uniform ball, of mass M = 6.00 kg and radius R, rolls smoothly from rest down a ramp at angle Θ = 30.0°.

a) The ball descends a vertical height h=1.20 m to reach the bottom of the ramp. What is its speed at the bottom?

Calculations: Where I_{com} is the ball's rotational inertia about an axis through its center of mass, v_{com} is the requested speed at the bottom, and w is the angular speed there.
Substituting, v_{com}/R for ω, and 2/5 MR^2 for I_{com},:

$$v_{com} = \sqrt{\left(\tfrac{10}{7}\right)gh} = \sqrt{\left(\tfrac{10}{7}\right)(9.8\text{ m/s}^2)(1.20\text{ m})}$$
$$= 4.10\text{ m/s.} \qquad \text{(Answer)}$$

b) What are the magnitude and direction of the frictional force on the ball as it rolls down the ramp?

Calculations: First we need to determine the ball's acceleration $a_{com,x}$:

$$a_{com,x} = -\frac{g\sin\theta}{1 + I_{com}/MR^2} = -\frac{g\sin\theta}{1 + \frac{2}{5}MR^2/MR^2}$$
$$= -\frac{(9.8\text{ m/s}^2)\sin 30.0°}{1 + \frac{2}{5}} = -3.50\text{ m/s}^2.$$

We can now solve for the frictional force:

$$f_s = -I_{com}\frac{a_{com,x}}{R^2} = -\tfrac{2}{5}MR^2\frac{a_{com,x}}{R^2} = -\tfrac{2}{5}Ma_{com,x}$$
$$= -\tfrac{2}{5}(6.00\text{ kg})(-3.50\text{ m/s}^2) = 8.40\text{ N.} \qquad \text{(Answer)}$$

Clicker question

A hollow ball and a solid ball roll without slipping down an inclined plane. Which ball reaches the bottom of the incline first?

A. The solid ball reaches the bottom first.
B. The hollow ball reaches the bottom first.
C. Both balls reach the bottom at the same time.
D. We can't determine this without information about the mass.

11.5: The Yo-Yo

1. Instead of rolling down a ramp at angle θ with the horizontal, the yo-yo rolls down a string at angle $\theta = 90°$ with the horizontal.
2. Instead of rolling on its outer surface at radius R, the yo-yo rolls on an axle of radius R_0 (Fig. 11-9*a*).
3. Instead of being slowed by frictional force $\vec{f}_s$, the yo-yo is slowed by the force $\vec{T}$ on it from the string (Fig. 11-9*b*).

The analysis would again lead us to Eq. 11-10. Therefore, let us just change the notation in Eq. 11-10 and set $\theta = 90°$ to write the linear acceleration as

$$a_{com} = -\frac{g}{1 + I_{com}/MR_0^2}, \qquad (11\text{-}13)$$

where I_{com} is the yo-yo's rotational inertia about its center and M is its mass. A yo-yo has the same downward acceleration when it is climbing back up.

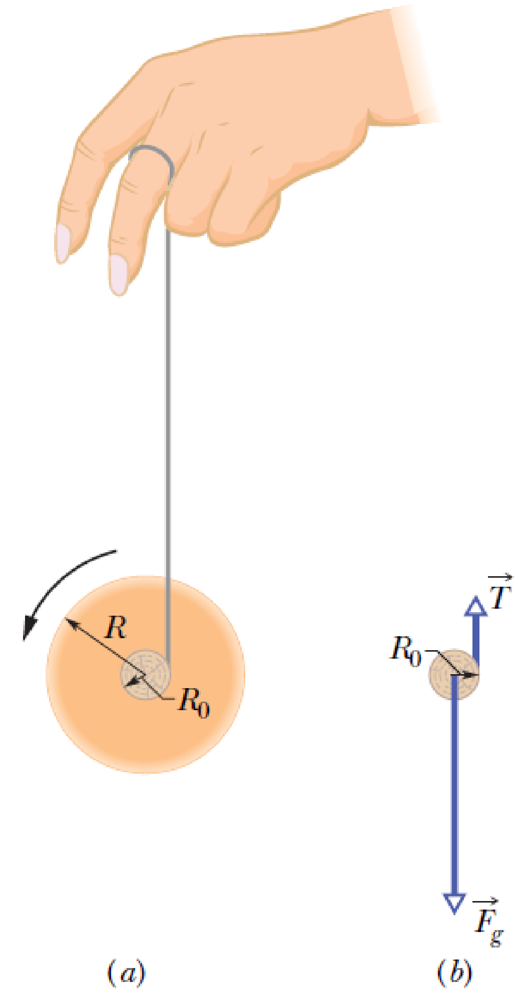
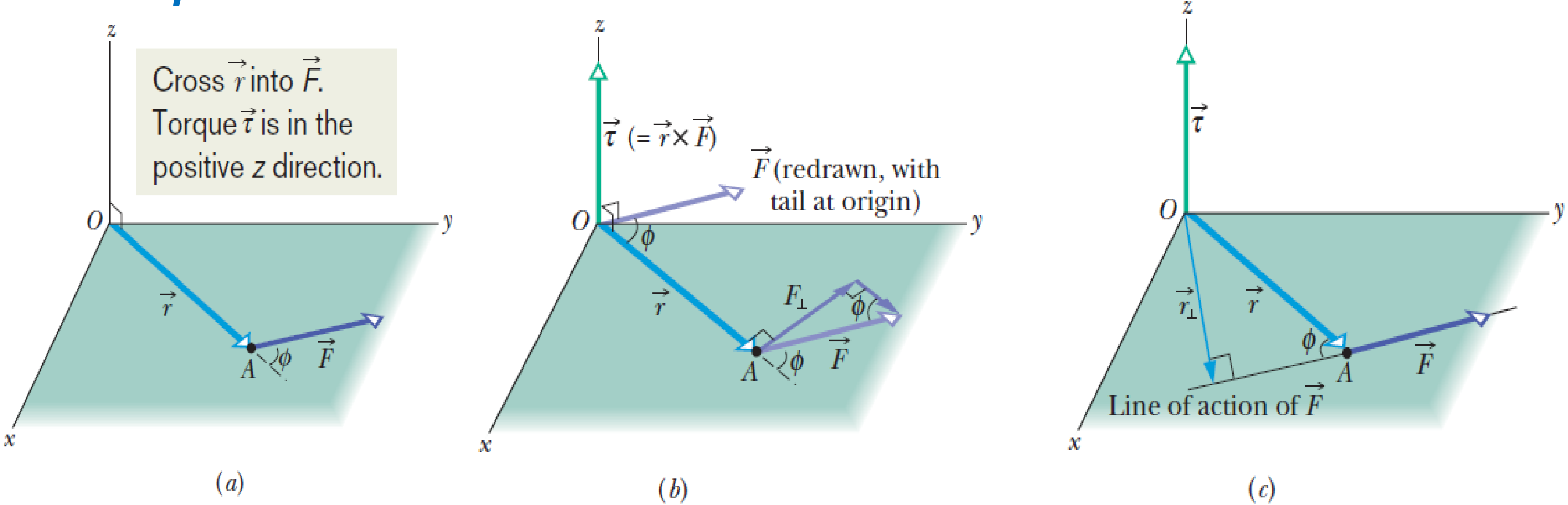


Fig. 11-9 (*a*) A yo-yo, shown in cross section. The string, of assumed negligible thickness, is wound around an axle of radius R_0. (*b*) A free-body diagram for the falling yo-yo. Only the axle is shown.

11.6: Torque Revisited



$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{(torque defined).}$$

$$\boxed{= rF \sin \phi,}$$

Fig. 11-10 *(a) A force F, lying in an x-y plane, acts on a particle at point A. (b) This force produces a torque* ***τ = r x F*** *on the particle with respect to the origin O. By the right-hand rule for vector (cross) products, the torque vector points in the positive direction of z. Its magnitude is given by $rF_\perp$ in (b) and by $r_\perp F$ in (c).*

3.8: Multiplying vectors: Vector (Cross) Product

The vector product between two vectors **a** and **b** can be written as:

$$\vec{a} \times \vec{b}$$

The result is a new vector **c**, which is:

$$c = ab \sin \phi,$$

Here a and b are the magnitudes of vectors **a** and **b** respectively, and *Φ* is the smaller of the two angles between **a** and **b** vectors.

The right-hand rule allows us to find the direction of vector c.

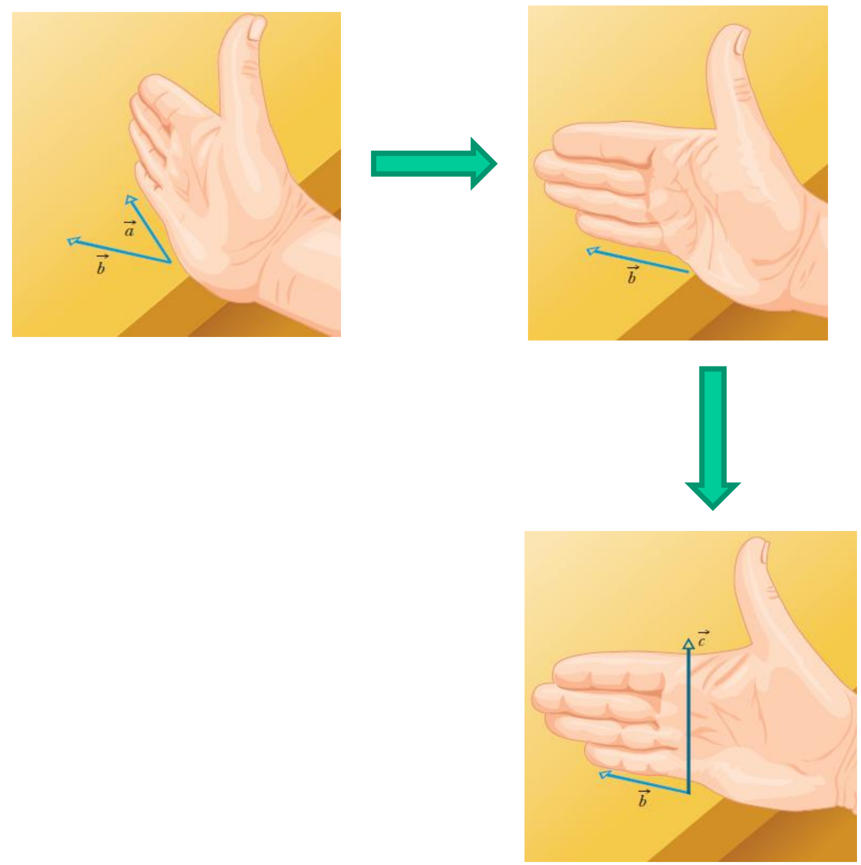


Fig. 3-19 Illustration of the right-hand rule for vector products. (*a*) Sweep vector $\vec{a}$ into vector $\vec{b}$ with the fingers of your right hand. Your outstretched thumb shows the direction of vector $\vec{c} = \vec{a} \times \vec{b}$.

3.8: Multiplying vectors: Vector product in unit-vector notation:

$$\vec{a} \times \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \times (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}})$$

$$= (a_y b_z - b_y a_z)\hat{\mathrm{i}} + (a_z b_x - b_z a_x)\hat{\mathrm{j}} + (a_x b_y - b_x a_y)\hat{\mathrm{k}}.$$

Note that: $$a_x\hat{\mathrm{i}} \times b_x\hat{\mathrm{i}} = a_x b_x(\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0,$$

And, $$a_x\hat{\mathrm{i}} \times b_y\hat{\mathrm{j}} = a_x b_y(\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y\hat{\mathrm{k}}$$

Example: Vector product

In Fig. 3-20, vector $\vec{a}$ lies in the xy plane, has a magnitude of 18 units and points in a direction 250° from the positive direction of the x axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the z axis. What is the vector product $\vec{c} = \vec{a} \times \vec{b}$?

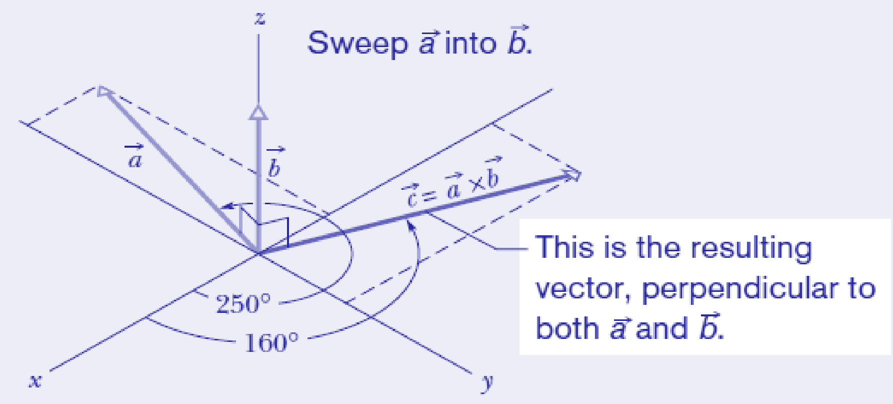


Fig. 3-20 Vector $\vec{c}$ (in the xy plane) is the vector (or cross) product of vectors $\vec{a}$ and $\vec{b}$.

KEY IDEA

When we have two vectors in magnitude-angle notation, we find the magnitude of their cross product with Eq. 3-27 and the direction of their cross product with the right-hand rule of Fig. 3-19.

Calculations: For the magnitude we write

$$c = ab \sin \phi = (18)(12)(\sin 90°) = 216. \qquad \text{(Answer)}$$

To determine the direction in Fig. 3-20, imagine placing the fingers of your right hand around a line perpendicular to the plane of $\vec{a}$ and $\vec{b}$ (the line on which $\vec{c}$ is shown) such that your fingers sweep $\vec{a}$ into $\vec{b}$. Your outstretched thumb then

gives the direction of $\vec{c}$. Thus, as shown in the figure, $\vec{c}$ lies in the xy plane. Because its direction is perpendicular to the direction of $\vec{a}$ (a cross product always gives a perpendicular vector), it is at an angle of

$$250° - 90° = 160° \qquad \text{(Answer)}$$

from the positive direction of the x axis.

Example: Vector product; unit vector notation

If $\vec{a} = 3\hat{\mathrm{i}} - 4\hat{\mathrm{j}}$ and $\vec{b} = -2\hat{\mathrm{i}} + 3\hat{\mathrm{k}}$, what is $\vec{c} = \vec{a} \times \vec{b}$?

KEY IDEA

When two vectors are in unit-vector notation, we can find their cross product by using the distributive law.

Calculations: Here we write

$$\begin{aligned}\vec{c} &= (3\hat{\mathrm{i}} - 4\hat{\mathrm{j}}) \times (-2\hat{\mathrm{i}} + 3\hat{\mathrm{k}})\\ &= 3\hat{\mathrm{i}} \times (-2\hat{\mathrm{i}}) + 3\hat{\mathrm{i}} \times 3\hat{\mathrm{k}} + (-4\hat{\mathrm{j}}) \times (-2\hat{\mathrm{i}})\\ &\quad + (-4\hat{\mathrm{j}}) \times 3\hat{\mathrm{k}}.\end{aligned}$$

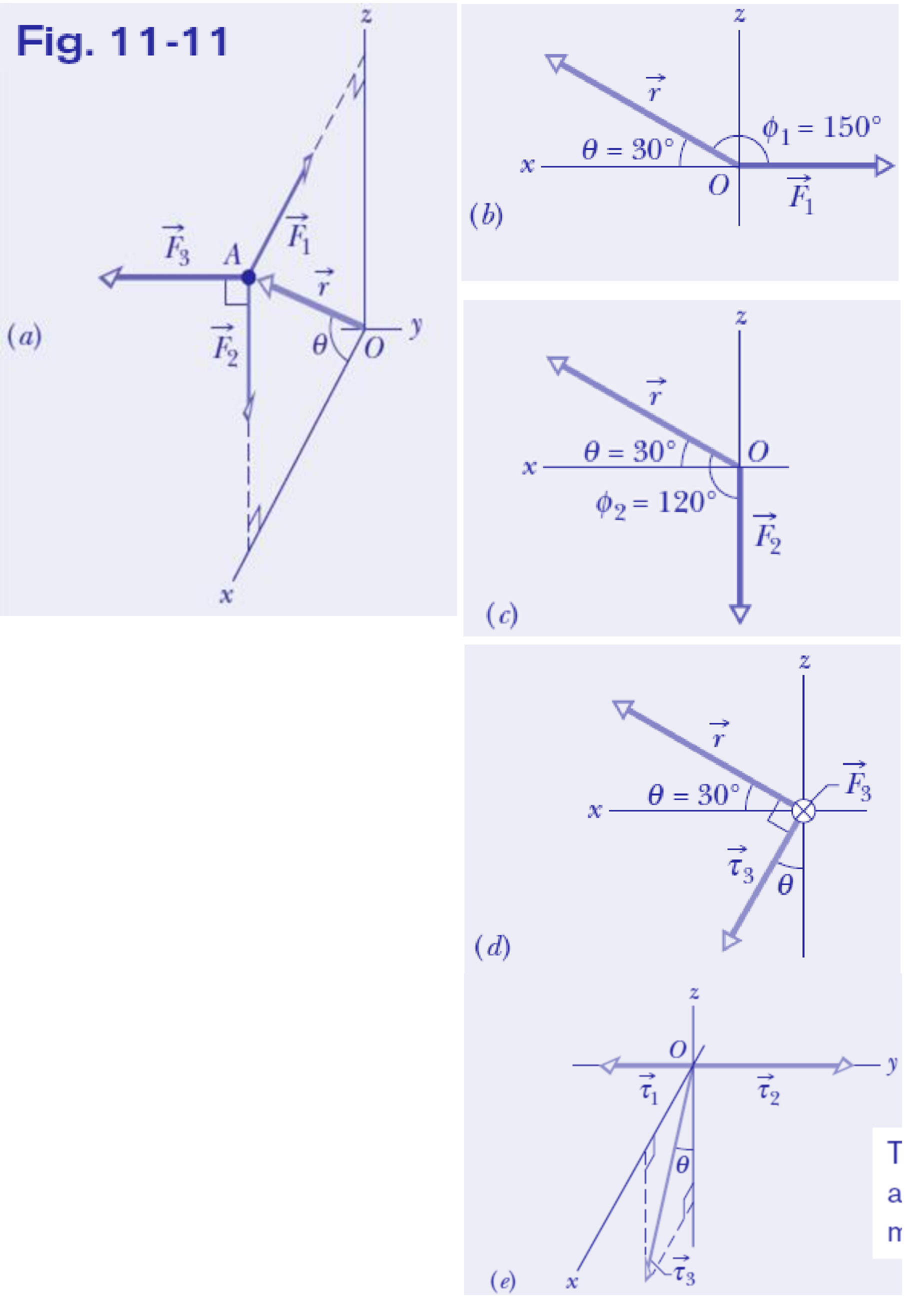
We next evaluate each term with Eq. 3-27, finding the direction with the right-hand rule. For the first term here, the angle ϕ between the two vectors being crossed is 0. For the other terms, ϕ is 90°. We find

$$\begin{aligned}\vec{c} &= -6(0) + 9(-\hat{\mathrm{j}}) + 8(-\hat{\mathrm{k}}) - 12\hat{\mathrm{i}}\\ &= -12\hat{\mathrm{i}} - 9\hat{\mathrm{j}} - 8\hat{\mathrm{k}}. \qquad \text{(Answer)}\end{aligned}$$

This vector $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, a fact you can check by showing that $\vec{c}\cdot\vec{a} = 0$ and $\vec{c}\cdot\vec{b} = 0$; that is, there is no component of $\vec{c}$ along the direction of either $\vec{a}$ or $\vec{b}$.

Example

Fig. 11-11



Calculations: Because we want the torques with respect to the origin *O*, the vector required for each cross product is the given position vector **r**.

To determine the angle θ between the direction of **r** and the direction of each force, we shift the force vectors of Fig. 11-11, each in turn, so that their tails are at the origin.

Figures b, c, and d, which are direct views of the *xz* plane, show the shifted force vectors $\mathbf{F_1}$, $\mathbf{F_2}$, and $\mathbf{F_3}$, respectively.

In Fig. d, the angle between the directions of and is 90°. Now, we find the magnitudes of the torques to be:

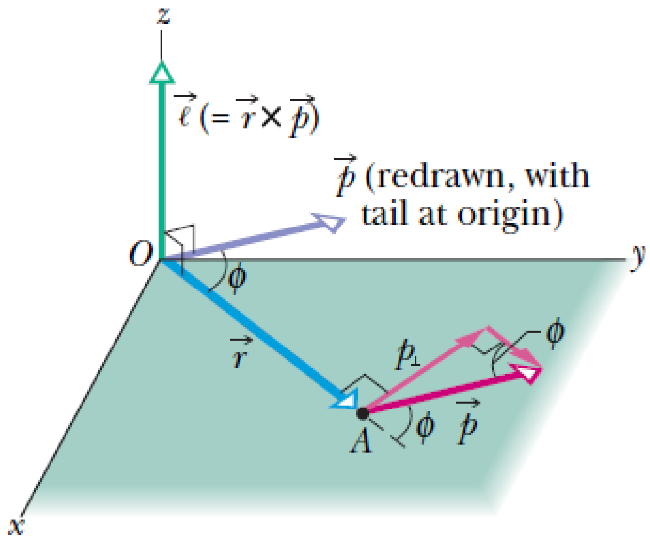
$$\tau_1 = rF_1 \sin\phi_1 = (3.0\text{ m})(2.0\text{ N})(\sin 150°) = 3.0\text{ N}\cdot\text{m},$$

$$\tau_2 = rF_2 \sin\phi_2 = (3.0\text{ m})(2.0\text{ N})(\sin 120°) = 5.2\text{ N}\cdot\text{m},$$

$$\tau_3 = rF_3 \sin\phi_3 = (3.0\text{ m})(2.0\text{ N})(\sin 90°)$$

$$= 6.0\text{ N}\cdot\text{m}. \qquad \text{(Answer)}$$

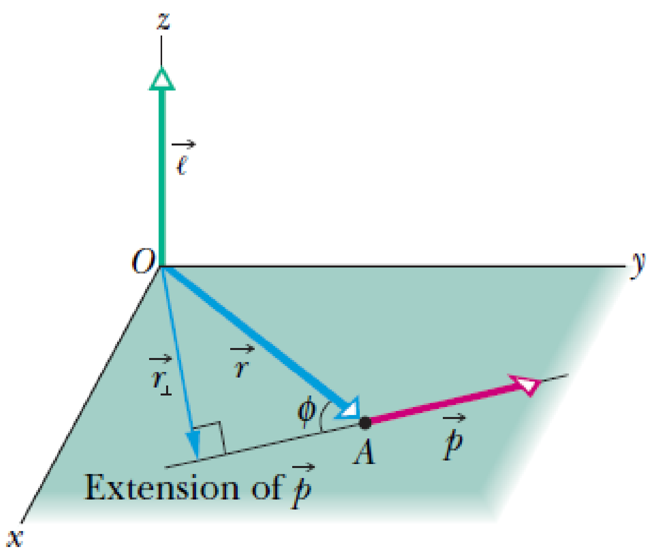
11.7 Angular Momentum



(*a*)

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) \qquad \text{(angular momentum defined),}$$

$$\ell = rmv \sin \phi \;= rp_\perp = rmv_\perp \;= r_\perp p = r_\perp mv$$



(*b*)

Fig. 11-12 Defining angular momentum. A particle passing through point A has linear momentum $\vec{p}\,(= m\vec{v})$, with the vector $\vec{p}$ lying in an xy plane. The particle has angular momentum $\vec{\ell}\,(= \vec{r} \times \vec{p})$ with respect to the origin O. By the right-hand rule, the angular momentum vector points in the positive direction of z. (*a*) The magnitude of $\vec{\ell}$ is given by $\ell = rp_\perp = rmv_\perp$. (*b*) The magnitude of $\vec{\ell}$ is also given by $\ell = r_\perp p = r_\perp mv$.

Example: Angular Momentum

Figure 11-13 shows an overhead view of two particles moving at constant momentum along horizontal paths. Particle 1, with momentum magnitude $p_1 = 5.0\ \text{kg}\cdot\text{m/s}$, has position vector $\vec{r}_1$ and will pass 2.0 m from point O. Particle 2, with momentum magnitude $p_2 = 2.0\ \text{kg}\cdot\text{m/s}$, has position vector $\vec{r}_2$ and will pass 4.0 m from point O. What are the magnitude and direction of the net angular momentum $\vec{L}$ about point O of the two-particle system?

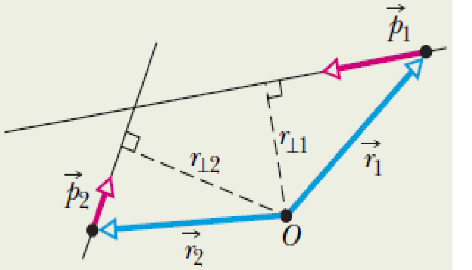


Fig. 11-13 Two particles pass near point O.

KEY IDEA

To find $\vec{L}$, we can first find the individual angular momenta $\vec{\ell}_1$ and $\vec{\ell}_2$ and then add them. To evaluate their magnitudes, we can use any one of Eqs. 11-18 through 11-21. However, Eq. 11-21 is easiest, because we are given the perpendicular distances $r_{1\perp}$ (= 2.0 m) and $r_{2\perp}$ (= 4.0 m) and the momentum magnitudes p_1 and p_2.

Calculations: For particle 1, Eq. 11-21 yields

$$\begin{aligned}\ell_1 &= r_{\perp 1} p_1 = (2.0\ \text{m})(5.0\ \text{kg}\cdot\text{m/s}) \\ &= 10\ \text{kg}\cdot\text{m}^2/\text{s}.\end{aligned}$$

To find the direction of vector $\vec{\ell}_1$, we use Eq. 11-18 and the right-hand rule for vector products. For $\vec{r}_1 \times \vec{p}_1$, the vector product is out of the page, perpendicular to the plane of Fig. 11-13. This is the positive direction, consistent with the counterclockwise rotation of the particle's position vector $\vec{r}_1$ around O as particle 1 moves. Thus, the angular momentum vector for particle 1 is

$$\ell_1 = +10\ \text{kg}\cdot\text{m}^2/\text{s}.$$

Similarly, the magnitude of $\vec{\ell}_2$ is

$$\begin{aligned}\ell_2 &= r_{\perp 2} p_2 = (4.0\ \text{m})(2.0\ \text{kg}\cdot\text{m/s}) \\ &= 8.0\ \text{kg}\cdot\text{m}^2/\text{s},\end{aligned}$$

and the vector product $\vec{r}_2 \times \vec{p}_2$ is into the page, which is the negative direction, consistent with the clockwise rotation of $\vec{r}_2$ around O as particle 2 moves. Thus, the angular momentum vector for particle 2 is

$$\ell_2 = -8.0\ \text{kg}\cdot\text{m}^2/\text{s}.$$

The net angular momentum for the two-particle system is

$$\begin{aligned}L &= \ell_1 + \ell_2 = +10\ \text{kg}\cdot\text{m}^2/\text{s} + (-8.0\ \text{kg}\cdot\text{m}^2/\text{s}) \\ &= +2.0\ \text{kg}\cdot\text{m}^2/\text{s}. \qquad \text{(Answer)}\end{aligned}$$

The plus sign means that the system's net angular momentum about point O is out of the page.

11.8: Newton's 2nd Law in Angular Form

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

$$\vec{\ell} = m(\vec{r} \times \vec{v}),$$

$$\frac{d\vec{\ell}}{dt} = m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}\right)$$

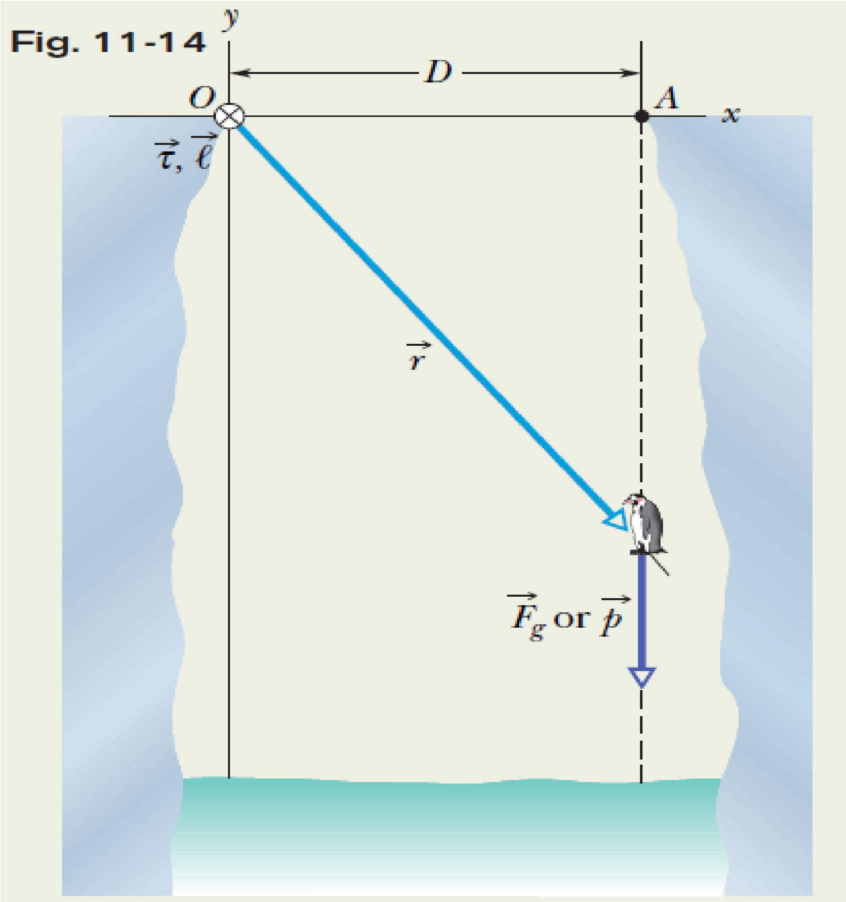
$$= m(\vec{r} \times \vec{a} + \vec{v} \times \vec{v}).$$

$$= m(\vec{r} \times \vec{a}) = \vec{r} \times m\vec{a}.$$

$$= \vec{r} \times \vec{F}_{\text{net}} = \sum(\vec{r} \times \vec{F}).$$

$$\boxed{\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt} \qquad \text{(single particle)}}$$

Example: Torque, Penguin Fall



In Fig. 11-14, a penguin of mass m falls from rest at point A, a horizontal distance D from the origin O of an xyz coordinate system. (The positive direction of the z axis is directly outward from the plane of the figure.)

(a) What is the angular momentum $\vec{\ell}$ of the falling penguin about O?

Calculations: The magnitude of l can be found by using:

$$\ell = r_{\perp} p = r_{\perp} m v$$

The perpendicular distance between O and an extension of vector **p** is the given distance D. The speed of an object that has fallen from rest for a time t is $v = gt$. Therefore,

$$\ell = r_{\perp} m v = Dmgt.$$

To find the direction of we use the right-hand rule for the vector product, and find that the direction is into the plane of the figure. The vector changes with time in magnitude only; its direction remains unchanged.

(b) About the origin O, what is the torque on the penguin due to the gravitational force ?

Calculations: $\tau = r_{\perp} F \Longrightarrow \tau = DF_g = Dmg$

Using the right-hand rule for the vector product we find that the direction of **t** is the negative direction of the z axis, the same as ***l***.

11.9: The Angular Momentum of a System of Particles

The total angular momentum **L** of the system is the (vector) sum of the angular momenta ***l*** of the individual particles (here with label *i):*

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i.$$

With time, the angular momenta of individual particles may change because of interactions between the particles or with the outside.

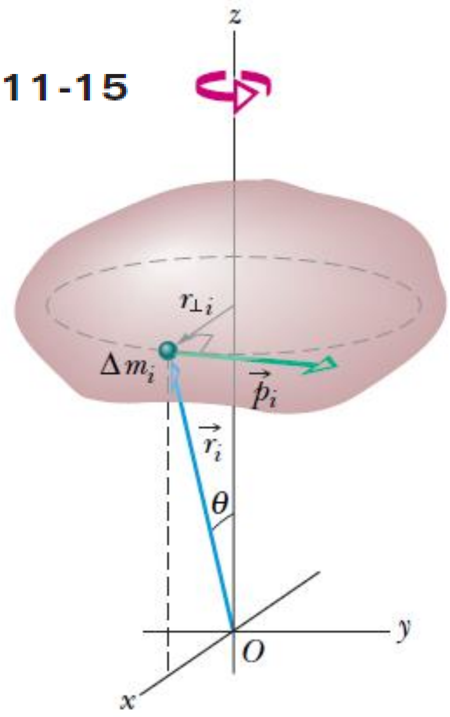
$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{\ell}_i}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\mathrm{net},i}.$$

Therefore, the net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum **L**.

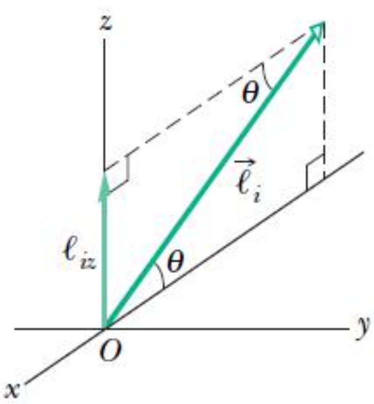
$$\vec{\tau}_{\mathrm{net}} = \frac{d\vec{L}}{dt} \qquad \text{(system of particles)}$$

11.10: Angular Momentum of a Rigid Body Rotating About a Fixed Axis

Fig. 11-15



(a)



(b)

a) A rigid body rotates about a z axis with angular speed w. A mass element of mass Δm_i within the body moves about the z axis in a circle with radius $r_{\perp i}$. The mass element has linear momentum p_i and it is located relative to the origin *O* by position vector r_i. Here the mass element is shown when $r_{\perp i}$ is parallel to the x axis.

b) The angular momentum l_i, with respect to O, of the mass element in (a). The z component ***l*** $_{iz}$ is also shown.

$$\ell_i = (r_i)(p_i)(\sin 90^\circ) = (r_i)(\Delta m_i\, v_i)$$

$$\ell_{iz} = \ell_i \sin\theta = (r_i \sin\theta)(\Delta m_i\, v_i) = r_{\perp i}\, \Delta m_i\, v_i.$$

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} \Delta m_i\, v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp i}) r_{\perp i}$$

$$= \omega \left(\sum_{i=1}^{n} \Delta m_i\, r_{\perp i}^2 \right).$$

$$L = I\omega \qquad \text{(rigid body, fixed axis).}$$

11.10: More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

[a]See also Table 10-3.

[b]For systems of particles, including rigid bodies.

[c]For a rigid body about a fixed axis, with L being the component along that axis.

[d]For a closed, isolated system.

11.11: Conservation of Angular Momentum

If the net external torque acting on a system is zero, the angular momentum **L** of the system remains constant, no matter what changes take place within the system.

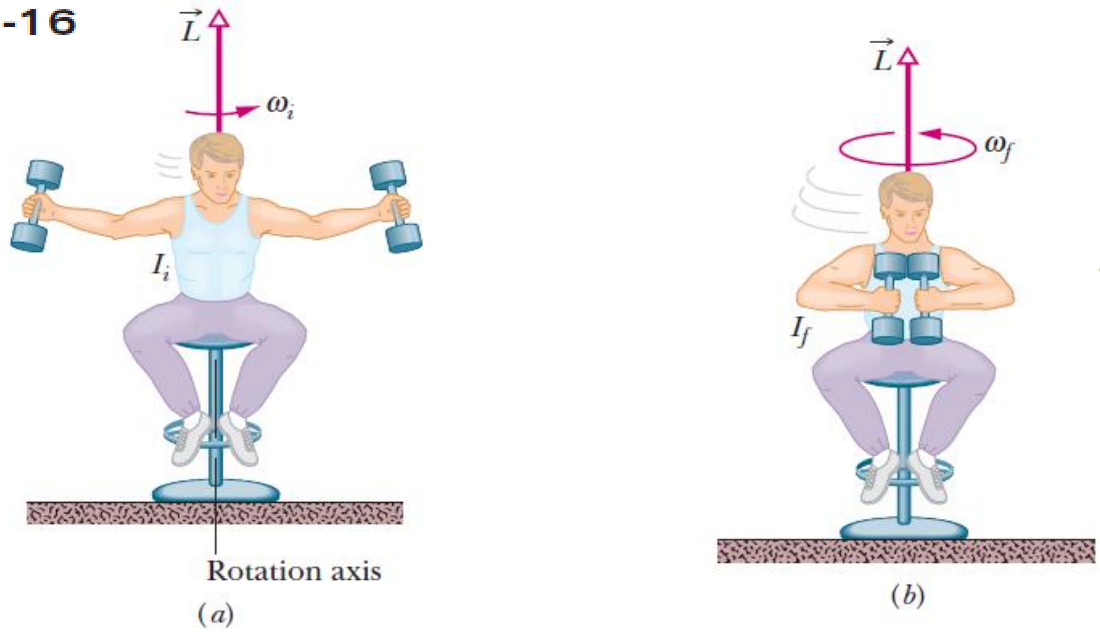
$$\vec{L} = \text{a constant} \qquad \text{(isolated system)}$$

$$\vec{L}_i = \vec{L}_f \qquad \text{(isolated system)}$$

11.11: Conservation of Angular Momentum

If the component of the net *external* torque on a system along a certain axis is zero, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

Fig. 11-16



$$I_i\omega_i = I_f\omega_f.$$

(a) The student has a relatively large rotational inertia about the rotation axis and a relatively small angular speed.

(b) By decreasing his rotational inertia, the student automatically increases his angular speed. The angular momentum of the rotating system remains constant or unchanged.

Example

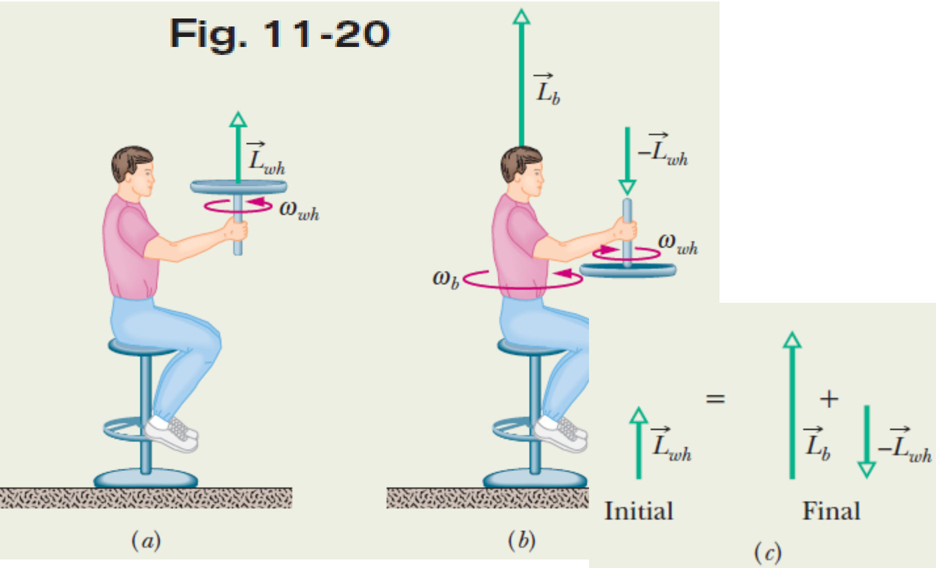


Figure 11-20 a shows a student, sitting on a stool that can rotate freely about a vertical axis. The student, initially at rest, is holding a bicycle wheel whose rim is loaded with lead and whose rotational inertia I_{wh} about its central axis is 1.2 kg m^2. (The rim contains lead in order to make the value of I_{wh} substantial.) The wheel is rotating at an angular speed ω_{wh} of 3.9 rev/s; as seen from overhead, the rotation is counterclockwise. The axis of the wheel is vertical, and the angular momentum L_{wh} of the wheel points vertically upward. The student now inverts the wheel (Fig. 11-20b) so L_{wh} that, as seen from overhead, it is rotating clockwise.

Its angular momentum is now - L_{wh}. The inversion results in the student, the stool, and the wheel's center rotating together as a composite rigid body about the stool's rotation axis, with rotational inertia I_b= 6.8 kg m^2. With what angular speed w_b and in what direction does the composite body rotate after the inversion of the wheel?

Calculations: The conservation of L_{tot} is represented with vectors in Fig. 11-20*c*. We can also write this conservation in terms of components along a vertical axis as

$$L_{b,f} + L_{wh,f} = L_{b,i} + L_{wh,i}, \tag{11-35}$$

where *i* and *f* refer to the initial state (before inversion of the wheel) and the final state (after inversion). Because inversion of the wheel inverted the angular momentum vector of the wheel's rotation, we substitute $-L_{wh,i}$ for $L_{wh,f}$. Then, if we set $L_{b,i} = 0$ (because the student, the stool, and the wheel's center were initially at rest), Eq. 11-35 yields

$$L_{b,f} = 2L_{wh,i}.$$

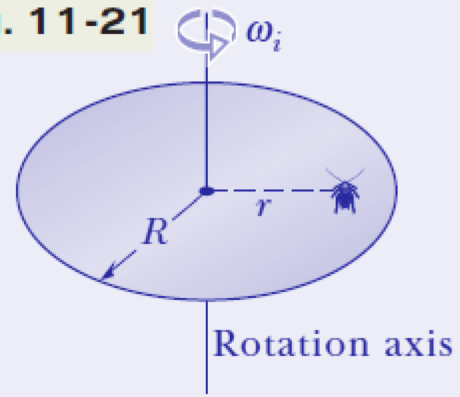
Using Eq. 11-31, we next substitute $I_b\omega_b$ for $L_{b,f}$ and $I_{wh}\omega_{wh}$ for $L_{wh,i}$ and solve for ω_b, finding

$$\omega_b = \frac{2I_{wh}}{I_b}\,\omega_{wh}$$

$$= \frac{(2)(1.2\ \text{kg}\cdot\text{m}^2)(3.9\ \text{rev/s})}{6.8\ \text{kg}\cdot\text{m}^2} = 1.4\ \text{rev/s}. \quad \text{(Answer)}$$

Example

Fig. 11-21



In Fig. 11-21, a cockroach with mass m rides on a disk of mass $6.00m$ and radius R. The disk rotates like a merry-go-round around its central axis at angular speed $\omega_i = 1.50$ rad/s. The cockroach is initially at radius $r = 0.800R$, but then it crawls out to the rim of the disk. Treat the cockroach as a particle. What then is the angular speed?

Calculations: We want to find the final angular speed. Our key is to equate the final angular momentum L_f to the initial angular momentum L_i, because both involve angular speed. They also involve rotational inertia I. So, let's start by finding the rotational inertia of the system of cockroach and disk before and after the crawl.

The rotational inertia of a disk rotating about its central axis is given by Table 10-2c as $\frac{1}{2}MR^2$. Substituting $6.00m$ for the mass M, our disk here has rotational inertia

$$I_d = 3.00mR^2. \tag{11-36}$$

(We don't have values for m and R, but we shall continue with physics courage.)

From Eq. 10-33, we know that the rotational inertia of the cockroach (a particle) is equal to mr^2. Substituting the cockroach's initial radius ($r = 0.800R$) and final radius ($r = R$), we find that its initial rotational inertia about the rotation axis is

$$I_{ci} = 0.64mR^2 \tag{11-37}$$

and its final rotational inertia about the rotation axis is

$$I_{cf} = mR^2. \tag{11-38}$$

So, the cockroach–disk system initially has the rotational inertia

$$I_i = I_d + I_{ci} = 3.64mR^2, \tag{11-39}$$

and finally has the rotational inertia

$$I_f = I_d + I_{cf} = 4.00mR^2. \tag{11-40}$$

Next, we use Eq. 11-31 ($L = I\omega$) to write the fact that the system's final angular momentum L_f is equal to the system's initial angular momentum L_i:

$$I_f\omega_f = I_i\omega_i$$

or $$4.00mR^2\omega_f = 3.64mR^2(1.50 \text{ rad/s}).$$

After canceling the unknowns m and R, we come to

$$\omega_f = 1.37 \text{ rad/s}. \qquad \text{(Answer)}$$

Note that the angular speed decreased because part of the mass moved outward from the rotation axis, thus increasing the rotational inertia of the system.

11.12: Precession of a Gyroscope

(a) A non-spinning gyroscope falls by rotating in an xz plane because of torque τ.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = Mgr \sin 90° = Mgr$$

(b) A rapidly spinning gyroscope, with angular momentum, **L**, precesses around the z axis. The precessional motion is in the *xy* plane.

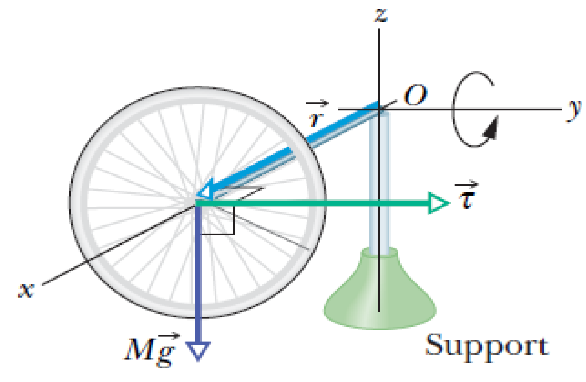
(c) The change in angular momentum, d**L**/dt, leads to a rotation of **L** about *O*.

$$\boxed{dL = \tau\, dt = Mgr\, dt.}$$

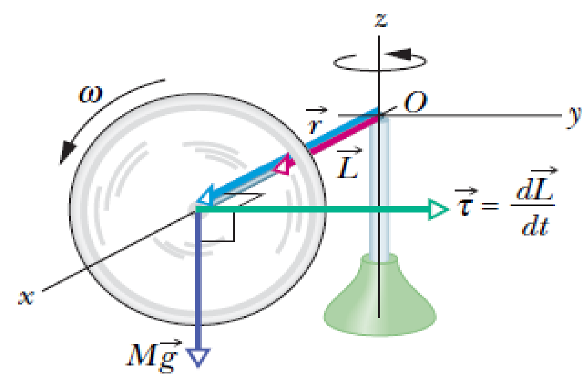
$$\boxed{d\phi = \frac{dL}{L} = \frac{Mgr\, dt}{I\omega}.}$$

$$\boxed{\Omega = d\phi/dt,}$$

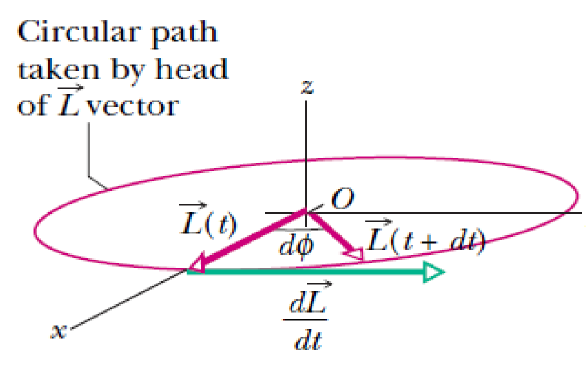
$$\Omega = \frac{Mgr}{I\omega} \qquad \text{(precession rate)}$$



(a)



(b)



(c)